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On a complex differential Riccati equation

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Abstract
We consider a nonlinear partial differential equation for complex-valued functions which is related to the two-dimensional stationary Schrödinger equation and enjoys many properties similar to those of the ordinary differential Riccati equation such as the famous Euler theorems, the Picard theorem and others. Besides these generalizations of the classical 'one-dimensional' results, we discuss new features of the considered equation including an analogue of the Cauchy integral theorem.

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1. Introduction

The ordinary differential Riccati equation

$$u' = pu^2 + qu + r \tag{1}$$

has received a great deal of attention since a particular version was first studied by Count Riccati in 1724, owing to both its specific properties and the wide range of applications in which it appears. For a survey of the history and classical results on this equation, see for example [7] and [25]. This equation can be always reduced to its canonical form (see, e.g., [6, 10])

$$y' + y^2 = v, \tag{2}$$

and this is the form that we will consider.

One of the reasons for which the Riccati equation has so many applications is that it is related to the general second-order homogeneous differential equation. In particular, the one-dimensional Schrödinger equation

$$-\partial^2 u + vu = 0 \tag{3}$$

is related to (2) by the easily inverted substitution

$$y = \frac{u'}{u}.$$

This substitution, which as its most spectacular application reduces Burger's equation to the standard one-dimensional heat equation, is the basis of the well-developed theory of logarithmic derivatives for the integration of nonlinear differential equations [19]. A generalization of this substitution will be used in this work.

A second relation between the one-dimensional Schrödinger equation and the Riccati equation is as follows. The one-dimensional Schrödinger operator can be factorized in the form

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x) = -\left(\frac{\mathrm{d}}{\mathrm{d}x} + y(x)\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - y(x)\right) \tag{4}$$

if and only if (2) holds. This observation is a key to a vast area of research related to the factorization method (see, e.g., [22] and [23]) and to the Darboux transformation (see, e.g., [20, 24] and [26]). In the present work, we consider a result similar to (4) but already in a two-dimensional setting.

Among the peculiar properties of the Riccati equation two theorems of Euler stand out, dating from 1760. The first of these states that if a particular solution y_0 of the Riccati equation is known, the substitution $y = y_0 + z$ reduces (2) to a Bernoulli equation which in turn is reduced by the substitution $z = \frac{1}{u}$ to a first-order linear equation. Thus given a particular solution of the Riccati equation, it can be linearized and the general solution can be found in two integrations. The second of these theorems states that given two particular solutions y_0, y_1 of the Riccati equation, the general solution can be found in the form

$$y = \frac{k y_0 \exp\left(\int y_0 - y_1\right) - y_1}{k \exp\left(\int y_0 - y_1\right) - 1}, \tag{5}$$

where k is a constant. That is, given two particular solutions of (2), the general solution can be found in one integration.

Other interesting properties are those discovered by Weyr and Picard [7, 29]. The first is that given a third particular solution y_3, the general solution can be found without integrating. That is, an explicit combination of three particular solutions gives the general solution. The second is that given a fourth particular solution y_4, the cross ratio

$$\frac{(y_1 - y_2)(y_3 - y_4)}{(y_1 - y_4)(y_3 - y_2)}$$

is a constant. Thus the derivative of this ratio is zero, which holds if and only if the numerator of the derivative is zero, that is, if and only if

$$(y_1 - y_4)(y_3 - y_2)((y_1 - y_2)(y_3 - y_4))' - (y_1 - y_2)(y_3 - y_4)((y_1 - y_4)(y_3 - y_2))' = 0.$$

Dividing by $(y_1 - y_2)(y_3 - y_4)(y_1 - y_4)(y_3 - y_2)$, we see that Picard's theorem is equivalent to the statement

$$\frac{(y_1 - y_2)'}{y_1 - y_2} + \frac{(y_3 - y_4)'}{y_3 - y_4} - \frac{(y_1 - y_4)'}{y_1 - y_4} - \frac{(y_3 - y_2)'}{y_3 - y_2} = 0. \tag{6}$$

In the present work, we study the following equation:

$$\partial_{\overline{z}} Q + |Q|^2 = v, \tag{7}$$

where z is a complex variable, $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$, Q is a complex-valued function of z and v is a real-valued function. Note that this equation is different from the complex Riccati equation studied in dozens of works where it is supposed to have the form (1) with complex analytic coefficients p, q and r and a complex analytic solution u (see, e.g., [9]). We do not suppose analyticity of the complex functions involved and show that equation (7) unlike the equation considered in [9] is related to the two-dimensional stationary Schrödinger equation in a similar way as the ordinary Riccati and Schrödinger equations are related in dimension 1. Moreover, we establish generalizations of the Euler and Picard theorems and obtain some other results which are essentially two dimensional, for example an analogue of the Cauchy integral theorem for solutions of the complex Riccati equation (7).

Equation (7) first appeared in [14] as a reduction to a two-dimensional case of the spatial factorization of the stationary Schrödinger operator, which was studied in a quaternionic setting in [1, 3, 11–14] and later on using Clifford analysis in [2] and [8].

The ordinary Riccati equation is at the heart of many analytical and numerical approaches to problems involving the one-dimensional Schrödinger and Sturm–Liouville equations. Here we furnish a complete structural analogy between dimensions 1 and 2 regarding the relationship between the Schrödinger and the Riccati equations. Besides, the deep similarity between the ordinary Riccati equation and (7) strongly suggests that many known applications of the ordinary Riccati equation can be generalized to the two-dimensional situation and many new aspects such as theorem 12 will become manifest.

2. Some preliminary results on the stationary Schrödinger equation and a class of pseudoanalytic functions

We need the following definition. Consider the equation $\partial_{\bar{z}}\varphi = \Phi$ in the whole complex plane or in a convex domain, where $\Phi = \Phi_1 + i\Phi_2$ is a given complex-valued function whose real part Φ_1 and imaginary part Φ_2 satisfy the equation

$$\partial_y \Phi_1 - \partial_x \Phi_2 = 0; \tag{8}$$

we can then reconstruct φ up to an arbitrary real constant c in the following way:

$$\varphi(x, y) = 2\left(\int_{x_0}^{x} \Phi_1(\eta, y)\,\mathrm{d}\eta + \int_{y_0}^{y} \Phi_2(x_0, \xi)\,\mathrm{d}\xi\right) + c, \tag{9}$$

where (x_0, y_0) is an arbitrary fixed point in the domain of interest.

By $\overline{A}$ we denote the integral operator in (9):

$$\overline{A}[\Phi](x, y) = 2\left(\int_{x_0}^{x} \Phi_1(\eta, y)\,\mathrm{d}\eta + \int_{y_0}^{y} \Phi_2(x_0, \xi)\,\mathrm{d}\xi\right) + c.$$

Note that formula (9) can be easily extended to any simply connected domain by considering the integral along an arbitrary rectifiable curve Γ leading from (x_0, y_0) to (x, y):

$$\varphi(x, y) = 2\left(\int_{\Gamma} \Phi_1\,\mathrm{d}x + \Phi_2\,\mathrm{d}y\right) + c.$$

Thus if Φ satisfies (8), there exists a family of real-valued functions φ such that $\partial_{\bar{z}}\varphi = \Phi$, given by the formula $\varphi = \overline{A}[\Phi]$. In a similar way, we introduce the operator

$$A[\Phi](x, y) = 2\left(\int_{\Gamma} \Phi_1\,\mathrm{d}x - \Phi_2\,\mathrm{d}y\right) + c,$$

which is applicable to complex functions satisfying the condition

$$\partial_y \Phi_1 + \partial_x \Phi_2 = 0 \tag{10}$$

and corresponds to the operator ∂_z.

Let f denote a positive twice continuously differentiable function defined in a domain $\Omega \subset \mathbb{C}$. Consider the following Vekua equation:

$$W_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{W} \qquad \text{in} \qquad \Omega, \tag{11}$$

where the subindex $\overline{z}$ means the application of the operator $\partial_{\overline{z}}$ (analogously, with the aid of the subindex z we will denote the application of the operator ∂_z), W is a complex-valued function and $\overline{W} = C[W]$ is its complex conjugate function. As was shown in [14, 15] (see also [16–18]), equation (11) is closely related to the second-order equation of the form

$$(-\Delta + \nu)u = 0 \qquad \text{in} \qquad \Omega, \tag{12}$$

where $\nu = (\Delta f)/f$ and u are real-valued functions. In particular, the following statements are valid.

Theorem 1 [15]. *Let W be a solution of (11). Then $u = \operatorname{Re} W$ is a solution of (12) and $v = \operatorname{Im} W$ is a solution of the equation*

$$(-\Delta + \eta)v = 0, \tag{13}$$

where $\eta = 2\left(\frac{|\nabla f|}{f}\right)^2 - \nu$.

Theorem 2 [15]. *Let u be a solution of (12) in a simply connected domain Ω. Then the function*

$$v \in \ker\left(\Delta + \nu - 2\left(\frac{|\nabla f|}{f}\right)^2\right),$$

such that $W = u + \mathrm{i}v$ be a solution of (11), is constructed according to the formula

$$v = f^{-1}\overline{A}(\mathrm{i}f^2\partial_{\overline{z}}(f^{-1}u)). \tag{14}$$

It is unique up to an additive term cf^{-1}, where c is an arbitrary real constant.

Given $v \in \ker\left(\Delta + \nu - 2\left(\frac{|\nabla f|}{f}\right)^2\right)$, the corresponding $u \in \ker(\Delta - \nu)$ can be constructed as follows:

$$u = -f\overline{A}(\mathrm{i}f^{-2}\partial_{\overline{z}}(fv)) \tag{15}$$

up to an additive term cf.

Thus, the relation between (11) and (12) is very similar to that between the Cauchy–Riemann system and the Laplace equation. Moreover, choosing $\nu \equiv 0$ and $f \equiv 1$ we arrive at the well-known formulae from the classical complex analysis. Note that the potential η has the form of a potential obtained after the Darboux transformation but already in two dimensions.

For a Vekua equation of the form

$$W_{\overline{z}} = aW + b\overline{W},$$

where a and b are arbitrary complex-valued functions from an appropriate function space [28], a well-developed theory of Taylor and Laurent series in formal powers has been created (see [4, 5]). We recall that a formal power $Z^{(n)}(a, z_0; z)$ corresponding to a coefficient a and a centre z_0 is a solution of the Vekua equation satisfying the asymptotic formula

$$Z^{(n)}(a, z_0; z) \sim a(z - z_0)^n, \qquad z \to z_0. \tag{16}$$

For a rigorous definition, we refer to [4, 5]. The theory of Taylor and Laurent series in formal powers among other results contains the expansion and the Runge theorems as well as more precise convergence results [21] and a recently obtained simple algorithm [17] for explicit construction of formal powers for the Vekua equation of the form (11). In section 4, we show how this theory is applied for generalizing the second Euler theorem. For this we need the expansion theorem from [4]. For the Vekua equation of the form (11), this expansion theorem reads as follows (for more details, we refer the reader to [17]).

Theorem 3. *Let W be a regular solution of (11) defined for $|z-z_0|<R$. Then it admits a unique expansion of the form $W(z)=\sum_{n=0}^{\infty} Z^{(n)}(a_n, z_0; z)$, which converges normally for $|z-z_0|<R$.*

Another result which will be used in the present work (section 4) is a Cauchy-type integral theorem for the stationary Schrödinger equation. It was obtained in [14] with the aid of the pseudoanalytic function theory.

Theorem 4 (Cauchy's integral theorem for the Schrödinger equation)**.** *Let f be a nonvanishing solution of (12) in a domain Ω and u be another arbitrary solution of (12) in Ω. Then for every closed curve Γ situated in a simply connected subdomain of Ω,*

$$\mathrm{Re}\int_\Gamma \partial_z\left(\frac{u}{f}\right)\mathrm{d}z + \mathrm{i}\,\mathrm{Im}\int_\Gamma f^2\partial_z\left(\frac{u}{f}\right)\mathrm{d}z = 0. \tag{17}$$

3. The two-dimensional stationary Schrödinger equation and the complex Riccati equation

Consider the complex differential Riccati equation

$$\partial_{\bar z} Q + |Q|^2 = \frac{\nu}{4}, \tag{18}$$

where for convenience the factor 1/4 was included. We recall that ν is a real-valued function. Together with this equation, we consider the stationary Schrödinger equation (12),

$$(-\Delta + \nu)u = 0, \tag{19}$$

where u is real valued. Both equations are studied in a domain $\Omega \subset \mathbb{C}$.

Theorem 5. *Let u be a solution of (19). Then its logarithmic derivative*

$$Q = \frac{u_z}{u} \tag{20}$$

is a solution of (18).

Proof. It is only necessary to substitute (20) into (18). □

Remark 6. Any solution of equation (18) fulfils (10). Indeed, the imaginary part of (18) reads as follows:

$$\partial_y Q_1 + \partial_x Q_2 = 0.$$

Theorem 7. *Let Q be a solution of (18). Then the function*

$$u = \mathrm{e}^{A[Q]} \tag{21}$$

is a solution of (19).

Proof. Equation (19) can be written in the form

$$(4\partial_{\bar z}\partial_z - \nu)u = 0.$$

Taking u in the form (21) where Q is a solution of (18) and using the observation that

$$\partial_{\bar{z}}(A[Q]) = \overline{\partial_z(A[Q])} = \overline{Q},$$

we have

$$\partial_{\bar{z}}\partial_z u = \partial_{\bar{z}}(Q\,\mathrm{e}^{A[Q]}) = \mathrm{e}^{A[Q]}(\partial_{\bar{z}}Q + |Q|^2) = \frac{\nu}{4}u. \qquad \square$$

Observe that this theorem means that if Q is a solution of (18), then there exists a solution u of (19) such that (20) is valid. Theorems 5 and 7 are direct generalizations of the corresponding facts from the one-dimensional theory.

The following statement is a generalization of the one-dimensional factorization (4).

Theorem 8. *Given a complex function Q, for any real-valued twice continuously differentiable function φ, the following equality is valid:*

$$\begin{aligned}\tfrac{1}{4}(\Delta - \nu)\varphi &= (\partial_{\bar{z}} + QC)(\partial_z - QC)\varphi \\ &= (\partial_z + \overline{Q}C)(\partial_{\bar{z}} - \overline{Q}C)\varphi \end{aligned} \tag{22}$$

if and only if Q is a solution of the Riccati equation (18).

Proof. Consider

$$(\partial_{\bar{z}} + QC)(\partial_z - QC)\varphi = \tfrac{1}{4}\Delta\varphi - |Q|^2\varphi - Q_{\bar{z}}\varphi$$

from which it is seen that (22) is valid iff Q is a solution of (18). The second equality in (22) is obtained by applying C to both sides of the first equality. $\square$

4. Generalizations of classical theorems

In this section, we give generalizations of both Euler's theorems for the Riccati equation as well as of Picard's theorem.

Theorem 9 (First Euler's theorem)**.** *Let Q_0 be a bounded particular solution of (18). Then (18) reduces to the following first-order (real-linear) equation:*

$$W_{\bar{z}} = \overline{Q_0 W} \tag{23}$$

in the following sense. Any solution of (18) has the form

$$Q = \frac{\partial_z \operatorname{Re} W}{\operatorname{Re} W}$$

and vice versa; any solution of (23) can be expressed via a corresponding solution Q of (18) as follows:

$$W = \mathrm{e}^{A[Q]} + \mathrm{i}\mathrm{e}^{-A[Q_0]}\overline{A}[\mathrm{i}\mathrm{e}^{2A[Q_0]}\partial_{\bar{z}}\,\mathrm{e}^{A[Q-Q_0]}]. \tag{24}$$

Proof. Let Q_0 be a bounded solution of (18). Then by theorem 7, we have that there exists a nonvanishing real-valued solution f of (19) such that $Q_0 = f_z/f$. Hence, (23) has the form (11). Now, let Q be any solution of (18). Then again $Q = u_z/u$, where u is a solution of (19). According to theorem 2, u is a real part of a solution W of (11). Thus, we have proved the first part of the theorem.

Let $W = u + \mathrm{i}v$ be any solution of (23) ($u = \operatorname{Re} W$). Then u is a solution of (19) and by theorem 5 it can be represented in the form $u = \mathrm{e}^{A[Q]}$, where Q is a solution of (18). Then by theorem 2 (formula (14)), W has the form (24). $\square$

Thus, the Riccati equation (18) is equivalent to a main Vekua equation of the form (11).

In what follows we suppose that in the domain of interest Ω, there exists a bounded solution of (18).

Theorem 10 (Second Euler's theorem)**.** *Any solution* Q *of equation (18) defined for* $|z-z_0| < R$ *can be represented in the form*

$$Q = \frac{\sum_{n=0}^{\infty} Q_n \, \mathrm{e}^{A[Q_n]}}{\sum_{n=0}^{\infty} \mathrm{e}^{A[Q_n]}}, \tag{25}$$

where $\{Q_n\}_{n=0}^{\infty}$ *is the set of particular solutions of the Riccati equation (18) obtained as follows:*

$$Q_n(z) = \frac{\partial_z \operatorname{Re} Z^{(n)}(a_n, z_0, z)}{\operatorname{Re} Z^{(n)}(a_n, z_0, z)},$$

where $Z^{(n)}(a_n, z_0, z)$ *are formal powers corresponding to equation (23) and both series in (25) converge normally for* $|z - z_0| < R$.

Proof. By the first Euler theorem, we have

$$Q = \frac{\partial_z \operatorname{Re} W}{\operatorname{Re} W},$$

where W is a solution of (23). From theorem 3, we obtain

$$Q(z) = \frac{\partial_z \sum_{n=0}^{\infty} \operatorname{Re} Z^{(n)}(a_n, z_0; z)}{\sum_{n=0}^{\infty} \operatorname{Re} Z^{(n)}(a_n, z_0; z)}.$$

Every formal power $Z^{(n)}(a_n, z_0; z)$ corresponds to a solution of (18):

$$Q_n(z) = \frac{\partial_z \operatorname{Re} Z^{(n)}(a_n, z_0; z)}{\operatorname{Re} Z^{(n)}(a_n, z_0; z)}$$

or $\operatorname{Re} Z^{(n)}(a_n, z_0; z) = \mathrm{e}^{A[Q_n](z)}$ from where we obtain (25). □

In the next statement, we give a generalization of Picard's theorem in the form (6).

Theorem 11 (Picard's theorem)**.** *Let* Q_k, $k = 1, 2, 3, 4$, *be four solutions of (18). Then*

$$\frac{\partial_{\bar z}(Q_1 - Q_2) + 2\mathrm{i}\operatorname{Im}(\overline{Q}_1 Q_2)}{Q_1 - Q_2} + \frac{\partial_{\bar z}(Q_3 - Q_4) + 2\mathrm{i}\operatorname{Im}(\overline{Q}_3 Q_4)}{Q_3 - Q_4} - \frac{\partial_{\bar z}(Q_1 - Q_4) + 2\mathrm{i}\operatorname{Im}(\overline{Q}_1 Q_4)}{Q_1 - Q_4} - \frac{\partial_{\bar z}(Q_3 - Q_2) + 2\mathrm{i}\operatorname{Im}(\overline{Q}_3 Q_2)}{Q_3 - Q_2} = 0.$$

Proof. Obviously,

$$(\overline{Q}_1 + \overline{Q}_4) + (\overline{Q}_3 + \overline{Q}_2) - (\overline{Q}_1 + \overline{Q}_2) - (\overline{Q}_3 + \overline{Q}_4) = 0.$$

Multiplying each parenthesis by $1 = (Q_i - Q_j)/(Q_i - Q_j)$, we obtain the equality

$$\frac{(\overline{Q}_1 + \overline{Q}_4)(Q_1 - Q_4)}{(Q_1 - Q_4)} + \frac{(\overline{Q}_3 + \overline{Q}_2)(Q_3 - Q_2)}{(Q_3 - Q_2)} - \frac{(\overline{Q}_1 + \overline{Q}_2)(Q_1 - Q_2)}{(Q_1 - Q_2)} - \frac{(\overline{Q}_3 + \overline{Q}_4)(Q_3 - Q_4)}{(Q_3 - Q_4)} = 0.$$

Using

$$(\overline{Q}_i + \overline{Q}_j)(Q_i - Q_j) = \partial_{\bar z}(Q_j - Q_i) - \overline{Q}_i Q_j + Q_i \overline{Q}_j,$$

the result is obtained. □

5. Cauchy's integral theorem

Theorem 12 (Cauchy's integral theorem for the complex Riccati equation)**.** *Let Q_0 and Q_1 be bounded solutions of (18) in Ω. Then for every closed curve Γ lying in a simply connected subdomain of Ω,*

$$\operatorname{Re}\int_\Gamma (Q_1 - Q_0)\,\mathrm{e}^{A[Q_1-Q_0]}\,\mathrm{d}z + \mathrm{i}\operatorname{Im}\int_\Gamma (Q_1 - Q_0)\,\mathrm{e}^{A[Q_1+Q_0]}\,\mathrm{d}z = 0. \tag{26}$$

Proof. From theorem 7, we have that $f = \mathrm{e}^{A[Q_0]}$ and $u = \mathrm{e}^{A[Q_1]}$ are solutions of (19). Now, applying theorem 4 we obtain

$$\operatorname{Re}\int_\Gamma \partial_z\left(\frac{u}{f}\right)\mathrm{d}z + \mathrm{i}\operatorname{Im}\int_\Gamma f^2\partial_z\left(\frac{u}{f}\right)\mathrm{d}z = 0$$

for every closed curve Γ situated in a simply connected subdomain of Ω, which gives us the equality

$$\operatorname{Re}\int_\Gamma \partial_z(\mathrm{e}^{A[Q_1-Q_0]})\,\mathrm{d}z + \mathrm{i}\operatorname{Im}\int_\Gamma \mathrm{e}^{2A[Q_0]}\partial_z(\mathrm{e}^{A[Q_1-Q_0]})\,\mathrm{d}z = 0.$$

From this, we obtain the result. □

As a particular case let us analyse the Riccati equation (18) with $\nu \equiv 0$ which is related to the Laplace equation. If in (26) we assume that $Q_0 \equiv 0$, then (26) takes the following form:

$$\int_\Gamma Q_1\,\mathrm{e}^{A[Q_1]}\,\mathrm{d}z = 0.$$

This is obviously valid because if Q_1 is another bounded solution of the Riccati equation with $\nu \equiv 0$, then according to theorem 7 we have that $u = \mathrm{e}^{A[Q_1]}$ is a harmonic function and the last formula turns into the equality

$$\int_\Gamma u_z\,\mathrm{d}z = 0 \tag{27}$$

(u_z is analytic).

Now, if in (26) we assume that $Q_1 \equiv 0$, then (26) takes the form

$$\operatorname{Re}\int_\Gamma Q_0\,\mathrm{e}^{-A[Q_0]}\,\mathrm{d}z + \mathrm{i}\operatorname{Im}\int_\Gamma Q_0\,\mathrm{e}^{A[Q_0]}\,\mathrm{d}z = 0.$$

Rewriting this equality in terms of the harmonic function $f = \mathrm{e}^{A[Q_0]}$, we obtain

$$\operatorname{Re}\int_\Gamma \frac{f_z\,\mathrm{d}z}{f^2} + \mathrm{i}\operatorname{Im}\int_\Gamma f_z\,\mathrm{d}z = 0$$

which, taking (27) into account, becomes the equality

$$\operatorname{Re}\int_\Gamma \frac{f_z\,\mathrm{d}z}{f^2} = 0$$

or which is the same,

$$\operatorname{Re}\int_\Gamma \partial_z\left(\frac{1}{f}\right)\mathrm{d}z = 0. \tag{28}$$

This equality is a simple corollary of a complex version of the Green–Gauss theorem (see, e.g., [27, section 3.2]), according to which we have

$$\frac{1}{2\mathrm{i}}\int_\Gamma \partial_z\left(\frac{1}{f}\right)\mathrm{d}z = \int_\Omega \partial_{\bar z}\partial_z\left(\frac{1}{f}\right)\mathrm{d}x\,\mathrm{d}y.$$

For f real the right-hand side is real valued and we obtain (28).

6. Conclusions

We have shown that the stationary Schrödinger equation in a two-dimensional case is related to a complex differential Riccati equation which possesses many interesting properties similar to its one-dimensional prototype. Besides the generalizations of the famous Euler theorems, we have obtained the generalization of Picard's theorem and the Cauchy integral theorem for solutions of the complex Riccati equation. The theory of pseudoanalytic functions has been intensively used.

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